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Focused Ground-Penetrating Radar Backprojection Through a Lossy Interface

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PREFACE

This document was prepared for the Deputy Under Secretary of Defense (Science and Technology) and the U.S. Army Communications Electronics Command under a task titled “Assessment for Mine Detection.”

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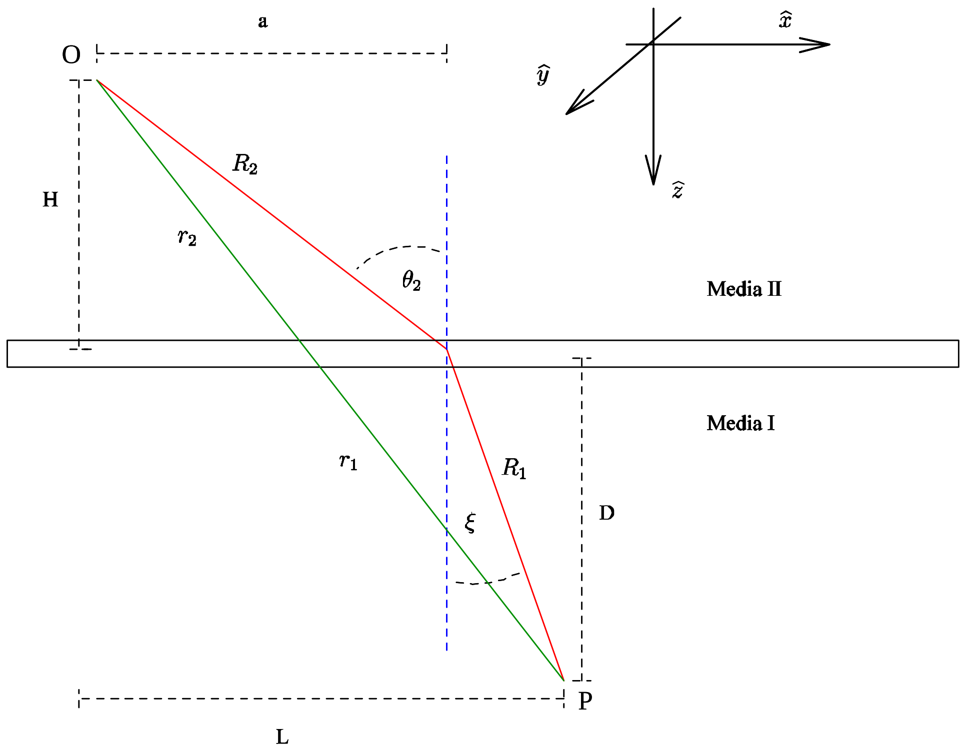

Figure 1. General GPR Geometry

To create a synthetic aperture radar (SAR) image using ground-penetrating radar (GPR) data, one must know the round-trip distance between the antenna aperture and the point to be focused on. Given a beginning point *O* in media II, a "buried" point *P* within media I, and the relationship between the points, the index of refraction *n* and the total "luminal" distance $R = n_1R_1 + n_2R_2$ must be derived (see Figure 1).[1] If we do not worry about refraction, the distance may be easily calculated using simple geometry. When refraction is taken into account, the problem becomes more difficult. When the refraction is governed by a lossy material, the problem becomes even more intractable and an appeal to numerical methods becomes necessary.

1 The total one-way travel time, T , for a ray of light is scaled by the speed of light in each respective media $T = (R_1/v_1) + (R_2/v_2)$. The physical situation may be interpreted as a scaling of the respective path lengths, $R = n_1 R_1 + n_2 R_2$, such that $T = R/c$.

A natural question is: Why would we be interested in accounting for refraction? There are two immediate advantages to doing so. First, correcting for refraction would deposit the energy received by the antenna in the correct image pixel (i.e., at the correct physical distance), thereby placing any image anomaly (i.e., a potential target of interest) at the correct depth. Second, *not* accounting for refraction would effectively blur the image and degrade image fidelity. The disadvantage is increased computational complexity and processing time. However, efficient software coding and carefully chosen numerical techniques can reduce the complexity and processing to an insignificant percentage of the total image-processing algorithm.

Deriving the equations that describe the optical path length between the originating point *O* and the terminating point *P* is best done through multiple transformations. The purpose of these transformations is to express the equations in a form easily understood with minimal mathematical complexity. The price to be paid is measured in proliferation of equations. However, at the end of the process, the resulting equations are few in number, comparatively simple, and written in terms of typical media parameters (dielectric constants, relative permeabilities, etc). Finally, Sections II.A and II.B closely follow the derivations presented in Stratton's *Electromagnetic Theory* [1] but with updated notation and use of magnitudes and phases instead of separation into real and imaginary parts.

II. SNELL'S LAW

A. COMPLEX ANGLES AND SNELL'S LAW

Given wave numbers $k_2 \in R$ and $k_1 \in C$, define $k_1 = |k_1|e^{\iota\chi}$. The standard form of Snell's Law, $k_1\sin\theta_1 = k_2\sin\theta_2$, will result in an imaginary θ_1:

$$\sin\theta_1 = \frac{k_2}{|k_1|} e^{-\iota\chi} \sin\theta_2 \tag{1}$$

$$\cos\theta_1 = \pm\sqrt{1 - \frac{k_2{}^2}{|k_1|^2} e^{-2\iota\chi} \sin^2\theta_2} \tag{2}$$

with the sign chosen by the boundary conditions and the requirement that the field is finite at all points in space. This is a problem because physically, the refraction angle cannot be complex. Moreover, we require a real geometric angle to focus SAR imagery. To facilitate our understanding of the relationship between the complex angle θ_1 and the physical refraction angle ξ, we expand cos θ_1 using complex notation such that

$$\cos\theta_1 = \rho e^{\iota\psi} \tag{3}$$

with a solution found for ρ and ψ by squaring and using (2). The solution to the above is straightforward

$$\rho^2 \cos 2\psi = 1 - \frac{k_2{}^2}{|k_1|^2} \sin^2\theta_2 \cos 2\chi \tag{4}$$

$$\rho^2 \sin 2\psi = 1 - \frac{k_2{}^2}{|k_1|^2} \sin^2\theta_2 \sin 2\chi \tag{5}$$

and it is trivial to show

$$\rho = \left(1 - 2\frac{k_2{}^2}{|k_1|^2} \sin^2\theta_2 \cos 2\chi + \frac{k_2{}^4}{|k_1|^4} \sin^4\theta_2\right)^{1/4} \tag{6}$$

$$\psi = \frac{1}{2}\arctan\left(\frac{k_2{}^2 \sin^2\theta_2 \sin 2\chi}{|k_1|^2 - k_2{}^2 \sin^2\theta_2 \cos 2\chi}\right). \tag{7}$$

B. ELECTROMAGNETIC WAVES IN BOTH MEDIA

We assume in media II an electromagnetic plane wave is present such that it intersects the boundary between lossless media II and lossy media I ($\chi \neq 0$) at an incident angle θ_2. Let E_I, E_R, and E_T be the magnitude of the incident, reflected, and transmitted electromagnetic fields, respectively. The total electromagnetic wave present will be

$$\text{II:} \qquad E_I e^{i\left(\vec{k}_2\cdot\vec{r}-\omega t\right)} + E_R e^{i\left(\vec{k}_2'\cdot\vec{r}-\omega t\right)}$$

$$\text{I:} \qquad E_T e^{i\left(\vec{k}_2\cdot\vec{r}-\omega t\right)}$$

in areas I and II. An electromagnetic wave propagates normal to the planes of constant phase, $\vec{k}\cdot\vec{r} = \textbf{constant}$. For all angles of incidence except $\theta_2 = 0$, the planes of constant phase will not necessarily be parallel to the planes of constant amplitude. Consequently, the field within medium I will not constitute a standard plane electromagnetic wave. For simplicity, align $\vec{k}_1 = k_1\left(\sin\theta_1\hat{x} + \cos\theta_1\hat{z}\right)$ such that

$$\begin{aligned}\vec{k}\cdot\vec{r}_1 &= \left|k_1\right|e^{\iota\chi}\left(\sin\theta_1\hat{x}+\cos\theta_1\hat{z}\right)\cdot\left(x\hat{x}+z\hat{z}\right)\text{then using (1)}\\ &= \mathrm{z}\left|\mathrm{k}_1\right|\rho e^{\iota(\chi+\psi)} + xk_2\sin\theta_2 \qquad .\\ &= \iota\left|k_1\right|\rho\sin\left(\chi+\psi\right)\mathrm{z} + \left|k_1\right|\rho\cos\left(\chi+\psi\right) + k_2\sin\theta_2 x\end{aligned} \tag{8}$$

Define

$$p\left(\theta_2\right) = p\left(\theta_2\right)\sin\left(\chi+\psi\left(\theta_2\right)\right) \tag{9}$$

$$q\left(\theta_2\right) = p\left(\theta_2\right)\cos\left(\chi+\psi\left(\theta_2\right)\right) \tag{10}$$

so the transmitted electromagnetic wave will be

$$\left(E_T e^{-\left|k_1\right|pz}\right)e^{\iota\left(\left|k_1\right|qz+k_2\sin\theta_2 x-\omega t\right)} \tag{11}$$

in lossy media I. Constant phase planes in media I will be defined by

$$z\cos\xi + x\sin\xi = \textbf{constant} \tag{12}$$

from which the true form of Snell's Law is revealed for lossy media

$$\sin\xi = \frac{k_2\sin\theta_2}{\sqrt{\left|k_1\right|^2 q\left(\theta_2\right)^2 + k_2^2\sin^2\theta_2}}\,. \tag{13}$$

We will find it profitable to explicitly note the above equation's "cousin":

$$\cos\xi = \frac{k_1 q\theta_2}{\sqrt{\left|k_1\right|^2 q\left(\theta_2\right)^2 + k_2^2\sin^2\theta_2}}\,. \tag{14}$$

Whereas the phase velocity of light in the lossless media is

$$\upsilon_2 = \frac{\omega}{k_2} \tag{15}$$

a careful look at (11) will reveal the phase velocity of light in the lossy media in our geometry of interest (i.e., two semi-infinite regions separated by a plane) is *dependent* on the angle of incidence,

$$\upsilon_1 = \frac{\omega}{\sqrt{|k_1|^2 q(\theta_2)^2 + k_2^2 \sin^2\theta_2}}\ . \tag{16}$$

We can define the ratio of the velocities as the ratio of the real index of refraction $n_1^+(\theta_2)$ in media I to the index of refraction n_2 in media II

$$\frac{n_1^+\theta_2}{n_2} \equiv \frac{\upsilon_2}{\upsilon_1} = \frac{\sin\theta_2}{\sin\xi} = \sqrt{\frac{|k_1|^2}{k_2^2} q(\theta_2)^2 + \sin^2\theta_2}\ . \tag{17}$$

After some manipulation we arrive at the following summarized results:

$$\rho(\theta_2) = \left[\left(1 - \frac{k_2^2}{|k_1|^2}\sin^2\theta_2\right)^2 + 4\frac{k_2^2}{|k_1|^2}\sin^2\theta_2\chi\right]^{1/4} \tag{18}$$

$$\psi(\theta_2) = \frac{1}{2}\arctan\left(\frac{k_2{}^2\sin^2\theta_2\sin 2\chi}{|k_1|^2 - k_2{}^2\sin^2\theta_2 2\chi}\right) \tag{19}$$

$$q(\theta_2) = \rho(\theta_2)\cos(\chi + \psi(\theta_2)) \tag{20}$$

$$\tan\xi = \frac{k_2\sin\theta_2}{|k_1| q(\theta_2)}\ . \tag{21}$$

C. SANITY CHECK: REDUCING TO STANDARD FORM

Let χ = 0, then (17)–(21) reduce to

$$\psi(\theta_2) = \frac{1}{2}\arctan(0) = 0 \tag{22}$$

$$\rho(\theta_2) = \left(1 - \frac{k_2^2}{k_1^2}\sin^2\theta_2\right)^{\frac{1}{2}} = \frac{1}{k_1}\sqrt{k_1^2 - k_2^2\sin^2\theta_2} \tag{23}$$

$$q(\theta_2) = \frac{1}{k_1}\sqrt{k_1^2 - k_2^2\sin^2\theta_2} \tag{24}$$

$$\frac{n_1^+(\theta_2)}{n_2} = \frac{k_1}{k_2} = \frac{n_1}{n_2}\,. \tag{25}$$

Given the above results and (13) we immediately see that

$$\sin\xi = \frac{k_2 \sin\theta_2}{\sqrt{|k_1|^2 q(\theta_2)^2 + k_2^2 \sin^2\theta_2}} \tag{26}$$

$$\Rightarrow \sin\xi = \frac{n_2}{n_1}\sin\theta_2 \tag{27}$$

which is the standard Snell's Law for refraction at the boundary between two lossless media.

D. RELATING WAVE NUMBERS TO TYPICAL PHYSICAL PARAMETERS

After reducing (17)–(21) to the standard form of Snell's law when $\chi = 0$, we relate k_1 and k_2 to typical known parameters. First, assume medium II is air so that $k_2 \approx \omega\sqrt{(\varepsilon_0\mu_0)} = \omega/c$. Second, there are a number of standard descriptions for a complex wave number. If medium I is completely described by a complex permittivity ε_1 and a real permeability μ_1, then

$$\begin{aligned} k_1 &= \omega\sqrt{\varepsilon_1\mu_1} = \omega\sqrt{|\varepsilon_1|\mu_1}\, e^{\frac{1}{2}\gamma} \\ |k_1| &= \omega\sqrt{|\varepsilon_1|\mu_1} \end{aligned} \tag{28}$$

$$\tan\chi = \tan\frac{1}{2}\gamma\,. \tag{29}$$

In general, the permittivity, permeability, and conductivity of medium I are all functions of frequency. When exact solutions are required, the appropriate $\varepsilon_1(\omega)$, $\mu_1(\omega)$, and/or $\sigma_1(\omega)$ must be inserted into (28)–(29). In practice, however, for many materials μ_1 is very near μ_0, and ε_1 and σ_1 are only weakly dependent on frequency.

The physical characteristics of many lossy materials are often described in the literature through the use of the relative (real) permeability $\mu_r = (\mu_1/\mu_0)$, the real dielectric constant $K = (\mathrm{Re}\{\varepsilon\}/\varepsilon_0)$, and the loss angle δ such that $\tan\delta = (\mathrm{Im}\{\varepsilon\}/\mathrm{Re}\{\varepsilon\})$. Therefore, the previous equations for $\rho(\theta_2)$ and $\psi(\theta_2)$ become

$$\rho(\theta_2) = \left[\left(1 - \frac{\sin^2\theta_2}{K\mu_r}\right)^2 + \sin^2\delta\right]^{1/4} \tag{30}$$

$$\psi(\theta_2) = \frac{1}{2}\arctan\left(\frac{\sin^2\theta_2 \sin 2\delta}{2k\mu_r - 2\sin^2\theta_2 \cos^2\delta}\right). \tag{31}$$

A little algebra[2] will provide

$$\cos\left(\frac{1}{2}\delta + \psi(\theta_2)\right) = \left[\frac{1}{2} + \frac{1}{2\sqrt{1 + \frac{\tan^2\delta}{\left(1 - \frac{\sin^2\theta_2}{K\mu_r}\right)^2}}}\right]^{\frac{1}{2}} \tag{32}$$

which, combined with the description of the lossy media (i.e., media I) using μ_r, K, and δ, describes the entire physical phenomena of refraction with the three equations:

$$q(\theta_2) = \left[\left(1 - \frac{\sin^2\theta_2}{K\mu_r}\right) + \sin^2\delta\right]^{1/4} \left[\frac{1}{2} + \frac{1}{2\sqrt{1 + \frac{\tan^2\delta}{\left(1 - \frac{\sin^2\theta_2}{K\mu_r}\right)^2}}}\right]^{1/2} \tag{33}$$

$$\tan\xi = \frac{\sin\theta_2}{q(\theta_2)}\sqrt{\frac{\cos\delta}{k\mu_r}} \tag{34}$$

$$\frac{n_1^+(\theta_2)}{n_2} = \sqrt{\frac{k\mu_r}{\cos\delta} q(\theta_2)^2 + \sin^2\theta_2}\ . \tag{35}$$

McLaurin expansion of (32) will give some physical insight as to the strength of lossy refraction compared with non-lossy refraction.

$$\cos\left(\frac{1}{2}\delta + \psi(\theta_2)\right) \cong \left[1 - \frac{1}{4}\frac{\tan^2\delta}{\left(1 - \frac{\sin^2\theta_2}{K\mu_r}\right)^2} + \frac{3}{16}\frac{\tan^4\delta}{\left(1 - \frac{\sin^2\theta_2}{K\mu_r}\right)^4} - \ldots\right]^{1/2} \tag{36}$$

---

2 See Appendix A for a short derivation.

where we immediately see $\cos(\delta/2 + \psi(\theta_2)) \cong 1$ and $\sin\xi = [\sin\theta_2/(\sqrt{K\mu_r})]$ for $\delta << \pi/4$. Therefore, materials with small loss angles can be treated as essentially lossless and the angle of refraction calculated from the familiar form of Snell's Law.

E. ILLUSTRATION BY EXAMPLE

For a soil mixture of 50% sand, 35% silt and 15% clay with a volumetric moisture content, m_v, of 0.05, experimental measurements give $K \approx 5$ and $\tan\delta \approx 1/10$ over the 300 MHz–1.3 GHz frequency range. Increasing the moisture content to $m_v = 0.20$ results in $K \approx 18$ and $\tan\delta \approx 2/18$ over the same frequency range [2]. In Figure 2 we see that the effect of a nonzero loss angle is extremely minor. For all values of the angle of incidence θ_2, the refraction angle is approximated extremely well by the standard form of Snell's law. At the limit $\theta_2 = \pi/2$, the difference between the case $K = 18$, $\tan\delta = 2/18$ and $K = 18$, $\tan\delta = 0$ is less than 4×10^{-4} radians. In Figure 3 we see the lossy material has an indices of refraction ratio weakly dependent on the angle of incidence.

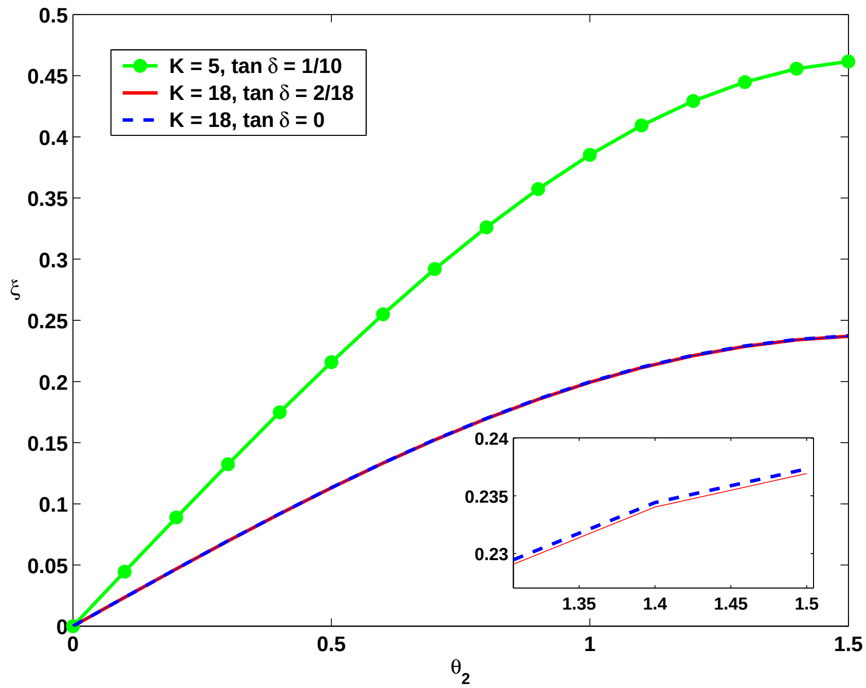

Figure 2. Snell's Law: Lossless versus Lossy Refraction

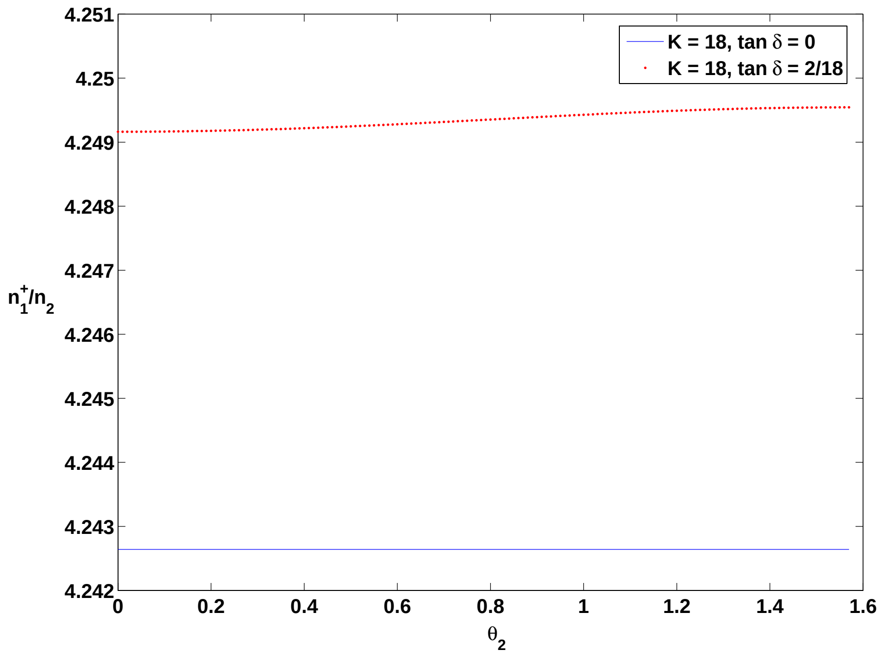

Figure 3. n_1^+/n_2 **: Lossless versus Lossy Refraction**

Therefore, as seen in Section II.D, materials with small loss angles, $\delta << \pi/4$, can be treated as essentially lossless. The ability to treat weakly lossy materials as lossless will greatly reduce the computational complexity of focusing GPR and simplify the relevant equations enough to allow for a closed-form solution.

Figure 4 illustrates an effect that will be of great interest to us in the following section, namely the limiting refraction angle 2ξ as the angle of incidence approaches the grazing angle:

$$\lim_{\theta_2 \to \frac{\pi}{2}} 2\xi \to 2\arctan\left(\frac{1}{q(\theta_2)}\sqrt{\frac{\cos\delta}{K\mu_r}}\right). \tag{37}$$

This effect will compete with the reduction in the wavelength, $\lambda = \lambda_0/n_1^+$, to determine the theoretical resolution of the GPR system. The figure plotted assumes the simplest case of $\mu_1 = \mu_0$ and frequency-independent K and δ. For small loss angles, the effect of absorption on the limiting angle of refraction is practically nonexistent. It is only for loss angles greater than approximately $\pi/4$ that the effect becomes noticeable. Evidently, the effect of the dielectric constant is significantly more important in determining the limiting refraction angle.

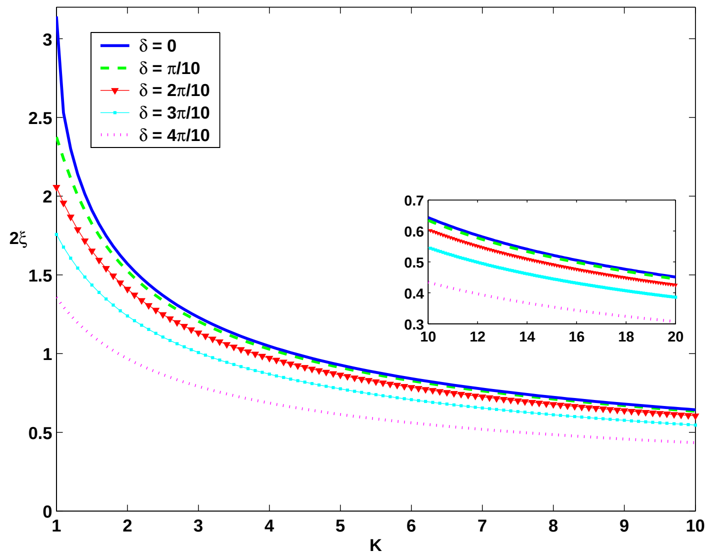

Figure 4. Refraction Limit 2ξ for $\theta_2 = \pi/2$

III. GPR BACKPROJECTION AND SNELL'S LAW

A. THEORETICAL SITUATION

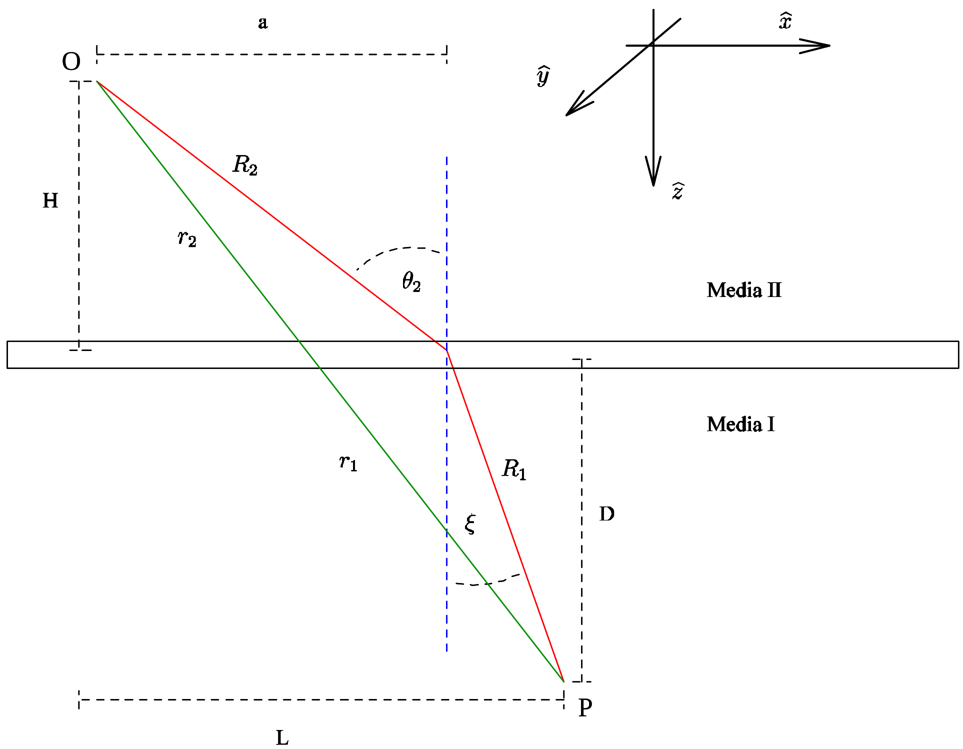

Figure 5. Theoretical Situation

Focused GPR backprojection requires knowledge of the round-trip time, $t = \left(n_1^{+} R_1 + n_2 R_2\right)/c$ between the points in the antenna aperture and the points in space we will focus on. We begin by assuming that we know both the coordinates of the antenna aperture O and the point P we wish to focus on: (x_2, y_2, z_2) and (x_1, y_1, z_1). We will further assume we have a right-handed coordinate frame where positive (increasing) values of z point into media I. For media II, we take the standard case of $n_2 \cong 1$ (air), and initially assume media I is a lossy soil mixture of 50% sand, 35% silt, and 15% clay (K = 5, tan δ = 1/10). However, because of the insignificant loss angle of many soils and the significantly increased computational complexity of treating the soil as lossy, we will eventually treat the soil as lossless and reap significant savings in processing time.

From Figure 5, standard geometry gives:

$$H = -z_2 \tag{38}$$

$$D = z_1 \tag{39}$$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{40}$$

$$R_2 = \frac{H}{\cos\theta_2} \tag{41}$$

$$\begin{gathered} R_1 = \frac{D}{\cos\xi} \\ \tan\xi = \frac{L-a}{D} = \frac{L - H\tan\theta_2}{D} \\ \frac{\sin\theta_2}{q(\theta_2)}\sqrt{\frac{\cos\delta}{K\mu_r}} = \frac{L - H\tan\theta_2}{D} \end{gathered} \tag{42}$$

from which we define (and require)

$$F(\theta_2, K, \delta, \mu_r D, L, H) = D\frac{\sin\theta_2}{q(\theta_2)}\sqrt{\frac{\cos\delta}{K\mu_r}} + H\tan\theta_2 - L = 0 \tag{43}$$

with the distance $n_2 R_2 + n_1^+ R_1$ given by

$$n_2 R_2 + n_1^+ R_1 = \frac{H}{\cos\theta_2} + \frac{D n_1^+}{\cos\xi} = \frac{H}{\cos\theta_2} + \frac{D}{q(\theta_2)}\sqrt{\frac{\cos\delta}{K\mu_r}}\left[\frac{K\mu_r}{\cos\delta}q(\theta_2)^2 + \sin^2\theta_2\right]. \tag{44}$$

Our task is to find the roots of (43) that lie within the space $\theta_2 \in [0, \pi/2]$, namely the angle of incidence given the coordinates of the origin and the point we wish to focus on. This will allow us to calculate our true interest, $n_1^+ R_1 + n_2 R_2$. The nature of $q(\theta_2)$ will make finding a closed-form solution difficult for lossy materials but, for lossless materials ($\delta = 0$), the solution to (43) can be written in closed form using the quartic formula to find the roots of the fourth-order polynomial:

$$\begin{aligned} &H^2(1 - K\mu_r)x^4 - 2HL(1 - K\mu_r)x^3 + \left[D^2 - H^2 K\mu_r + L^2(1 - K\mu_r)\right]x^2 \\ &+ 2HLK\mu_r x - L^2 K\mu_r \end{aligned} \tag{45}$$

where $\theta_2 = \arctan x$. In solving (45), care must be taken in choosing the correct root, $x \in [0, \infty]$. Since many soils are characterized by small loss angles, equation (45) can often be used in place of the significantly more complicated (43), resulting in decreased computational complexity.[3]

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[3] See Appendix B for further discussion.

B. NUMERICAL ANALYSIS

Assume that the synthesized aperture will focus at the point (0, 0, 3) and the antenna moves along a grid 5 cm above the surface. Using Matlab R14 we are able to solve for the angle of incidence given the aperture and "target" coordinates, and the results are illustrated in Figure 6 for our test soil mixture. The details of the figure match physical intuition: It is symmetric about the origin and disappears at the origin.

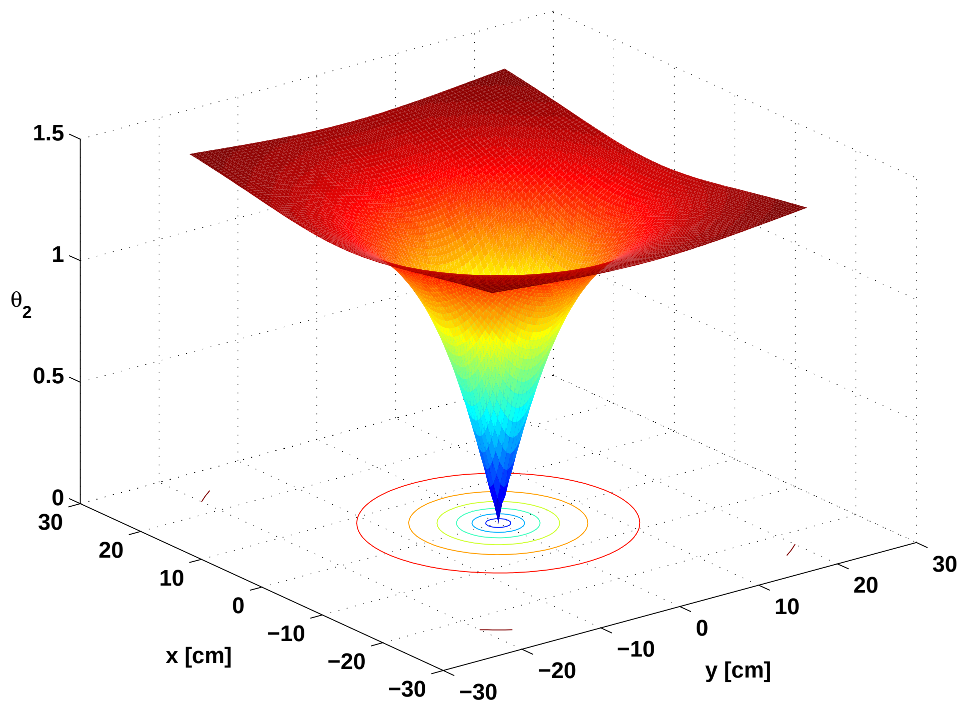

Figure 6. Angle of Incidence Given *P* at (0, 0, 3) and *O* at (*x, y,* –5)

To qualitatively understand how important refraction is in determining the luminal path length $R = n_1^+ R_1 + n_2 R_2$, we will measure the difference between the true path traveled and the geometric, straight-line path defined by $n_1 r_1 + n_2 r_2$.

$$n_1 r_1 + n_2 r_2 = \sqrt{\frac{(H+D)^2 + L^2}{H+D}}\left[D\sqrt{K\mu_r} + H\right]. \tag{46}$$

The maximum one-way difference between the luminal path length and the geometric path length is approximately 8 cm (or 0.27 ns) when the antenna phase center is at (–25, 0, –5) cm and is focusing at the point (0, 0, 7) cm. It is difficult to intuitively decide how much of an effect these differences would have on GPR data processing. On one hand, between 500 MHz–6 GHz an 8 cm difference will be approximately 0.13 to 1.6

wavelengths, which can introduce serious phase errors when summing the signal over the aperture. On the other hand, when the antenna is operated perpendicular and near the surface, most of the difference appears only at points where most of the signal received by the antenna would probably be of small amplitude compared with signals received from points with no significant need of refraction correction. However, antennas with main lobes of large angular extent or geometries where the antenna is oblique to the surface may increase the relative signal amplitudes. Therefore, the relative importance of correctly accounting for refraction can only be determined on a per antenna basis where Figure 7 is weighted by the antenna beam pattern and due consideration is given to the geometry of the physical system.

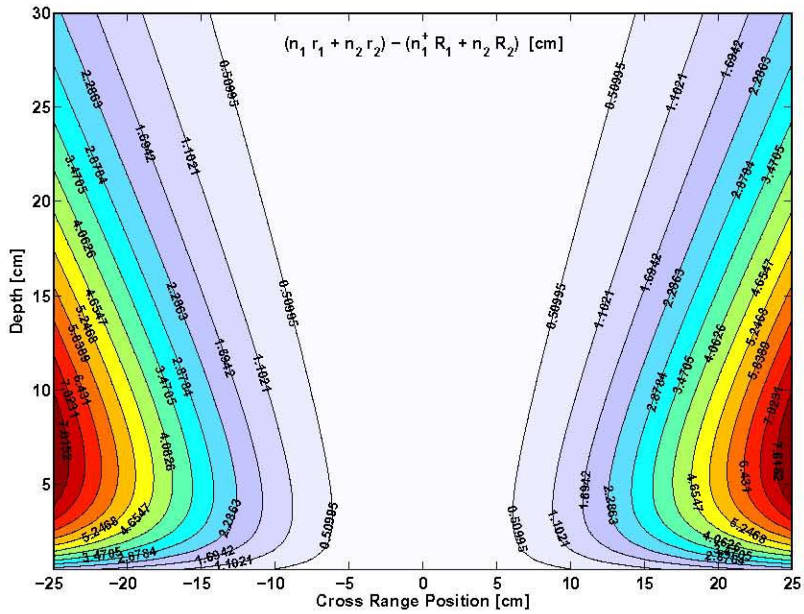

Figure 7. One-Way Difference with *P* at (0, 0, *d*) and *O* at (*x*, 0, –5)

IV. CONCLUSION

In summary, we have derived the Snell's Law equations describing the refractory behavior of electromagnetic waves intersecting a boundary between lossless media and lossy media under the assumption of the existence of electromagnetic plane waves in media II. Furthermore, using Snell's Law, we have explicitly derived the equations needed to describe the geometry of a standard GPR system-target package and quantitatively calculated the expected correction for a test case utilizing realistic soil parameters. However, the validity and urgency of correctly accounting for the refractive behavior is strongly dependent on the parameters of the GPR antenna utilized and the relative orientation of the antenna to the boundary.

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[1] Stratton, Julius A., *Electromagnetic Theory*, New York: McGraw-Hill, 1941.

[2] Peplinski, Neil R., Fawwaz T. Ulaby, Myron C. Dobson, “Dielectric Properties of Soils in the 0.3–1.3 GHz Range,” *IEEE Transactions on Geoscience and Remote Sensing*, Vol. 33, No. 3, pp. 803–807, May 1995.

APPENDIX

A. DERIVATION OF $\cos\left[\frac{1}{2}\delta + \psi(\theta_2)\right]$

Consider the square of the term we are interested in and expand

$$\cos^2 \frac{1}{2}[\delta + \psi] = \frac{1}{2} + \frac{1}{2}[\cos\delta \cos 2\psi - \sin\delta \sin 2\psi]. \tag{47}$$

Then from equation (31) we immediately have

$$\tan 2\psi = \frac{\sin 2\delta \sin^2\theta_2}{2K\mu_r - 2\cos^2\delta \sin^2\theta_2} \equiv \frac{A}{B} \tag{48}$$

and therefore

$$\cos 2\psi = \frac{B}{\sqrt{A^2 + B^2}} \tag{49}$$

$$\sin 2\psi = \frac{A}{\sqrt{A^2 + B^2}}\ . \tag{50}$$

Simple algebra gives

$$A^2 + B^2 = 4\cos^2\delta\left[\left(K\mu_r - \sin^2\theta_2\right)^2 + \left(K\mu_r\right)^2 \tan^2\delta\right] \tag{51}$$

from which we may easily see

$$\cos\delta \cos 2\psi = \frac{K\mu_r - \cos^2\delta \sin^2\theta_2}{\sqrt{\left(K\mu_r - \sin^2\theta_2\right)^2 + \left(K\mu_r\right)^2 \tan^2\delta}} \tag{52}$$

$$\sin\delta \sin 2\psi = \frac{\sin^2\delta \sin^2\theta_2}{\sqrt{\left(K\mu_r - \sin^2\theta_2\right)^2 + \left(K\mu_r\right)^2 \tan^2\delta}}\ . \tag{53}$$

Therefore,

$$\cos^2\left[\frac{1}{2}\delta + \psi\right] = \frac{1}{2} + \frac{1}{2}\frac{K\mu_r - \sin^2\theta_2}{\sqrt{\left(K\mu_r - \sin^2\theta_2\right)^2 + \left(K\mu_r\right)^2 \tan^2\delta}} \tag{54}$$

$$\rightarrow \cos\left[\frac{1}{2}\delta+\psi\right]\left[\frac{1}{2}+\frac{1}{2\sqrt{1+\dfrac{\tan^2\delta}{\left(1-\dfrac{\sin^2\theta_2}{K\mu_r}\right)^2}}}\right]^{1/2}. \tag{55}$$

B. QUARTIC POLYNOMIAL DESCRIPTION FOR LOSSLESS MATERIALS

If $\delta = 0$, then (43) reduces to

$$\frac{D\sin\theta_2}{q(\theta_2)}\frac{1}{\sqrt{K\mu_r}}+H\tan\theta_2-L=0 \tag{56}$$

$$\frac{D\sin\theta_2}{\sqrt{K\mu_r-\sin^2\theta}}+H\tan\theta_2-L=0\,. \tag{57}$$

Squaring both sides and replacing terms gives

$$\frac{D^2\sin^2\theta_2}{K\mu_r-\sin^2\theta_2}+\left(H\tan\theta_2-L\right)^2+2\left(H\tan\theta_2-L\right)\frac{D\sin\theta_2}{\sqrt{K\mu_r-\sin^2\theta_2}}=0 \tag{58}$$

$$D^2\sin^2\theta_2-\left(H\tan\theta_2-L\right)^2\left(K\mu_r-\sin^2\theta_2\right)=0\,. \tag{59}$$

It will be advantageous to make the substitution

$$\tan\theta_2 = x \tag{60}$$

$$\rightarrow \sin^2\theta=\frac{x^2}{x^2+1}\,. \tag{61}$$

Then after expansion and term collection, we have

$$D^2x^2-\left(Hx-L\right)^2\left[K\mu_r\left(x^2+1\right)-x^2\right]=0 \tag{62}$$

and we can easily derive the quartic polynomial

$$\begin{aligned} &H^2\left(1-K\mu_r\right)x^4-2HL\left(1-K\mu_r\right)x^3+\left[D^2-H^2K\mu_r+L^2\left(1-K\mu_r\right)\right]x^2 \\ &\quad +2HLK\mu_r x-L^2K\mu_r \end{aligned}\,. \tag{63}$$

Practical use of (63) will require care taken in choosing the correct root, $x\in[0,\infty)$. Although not explored in this paper, finite group theory may shed some light in understanding the behavior of the roots of (63) as a function of D, H, L, K, and μ_r.

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